Supersymmetric Solutions in Six Dimensions: A Linear Structure

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Abstract

The equations underlying all supersymmetric solutions of six-dimensional minimal ungauged supergravity coupled to an anti-self-dual tensor multiplet have been known for quite a while, and their complicated non-linear form has hindered all attempts to systematically understand and construct supersymmetric solutions. In this paper we show that, by suitably re-parameterizing these equations, one can find a structure that allows one to construct supersymmetric solutions by solving a sequence of linear equations. We then illustrate this method by constructing a new class of geometries describing several parallel spirals carrying D1, D5 and P charge and parameterized by four arbitrary functions of one variable. A similar linear structure is known to exist in five dimensions, where it underlies the black hole, black ring and corresponding microstate geometries. The unexpected generalization of this to six dimensions will have important applications to the construction of new, more general such geometries.

1 Introduction

Much of the remarkable progress in constructing and classifying three-charge, supersymmetric solutions in five-dimensions was made possible by the observation that the underlying equations [1, 2, 3], contrary to initial appearances, are *linear* [3]: the whole system can be reduced to solving the equations of four-dimensional Euclidean electromagnetism with sources. This enabled the discovery of the bubbling transition [4, 5], which lies at the core of the construction of supersymmetric black-hole microstate geometries, and has also led to rich new classes of BPS[1] solutions.

It is a natural and important question to ask whether this linearity extends to higher dimensional supergravities and, more particularly, to the six-dimensional supergravity theory [6, 7, 8, 9, 10, 11] whose reduction yields the five-dimensional, $\mathcal{N} = 2$ ungauged supergravity theory where the linear structure is known to exist. Given that all supersymmetric, asymptotically $AdS_3 \times S^3$ solutions that are dual to the D1-D5 system belong to this class, the significance of a linear structure in the six-dimensional theory can hardly be overemphasized. The generic BPS geometry in six dimensions would, of course, differ from its five-dimensional counterpart in that it can depend non-trivially upon the compactification direction and this could easily lead to non-linearities. On the other hand, if the equations were still linear it would not only make the construction of explicit solutions much more straightforward but it would also enable a much more precise analysis of the moduli space of such solutions.

One of the key aspects of the fuzzball proposal[2] is that the microstate geometries of a black hole constructed in four-dimensional supergravity cannot be properly described and counted unless one works in a higher dimensional supergravity. Indeed, the singularity theorems of four-dimensional general relativity tell us that there are essentially no four-dimensional microstate geometries. On the other hand, the structure of microstate geometries becomes far richer in five and six dimensions. The bubbling transition that resolves the singularity and gives the smooth microstates of the four-dimensional D6-D2-D2-D2 black hole naturally emerges in five-dimensional supergravity [4, 5, 18].

It is true that the easiest-to-construct five-dimensional microstate geometries have a tri-holomorphic $U(1)$ invariance, and thus descend to multi-center black-hole solutions in four-dimensions [19, 20]. Such microstate geometries may therefore be thought of as being four-dimensional but they are singular from the four-dimensional perspective. Moreover, solutions with this $U(1)$ invariance have a finite-dimensional moduli space, and hence are "too rigid" to account for entropy of the black hole [21, 22, 23][3]. Typical microstates will not have this $U(1)$ invariance and hence will be intrinsically five-dimensional, at least. Indeed, a lot of recent work suggests that, while the essential geometric transition can be made in five dimensions, it is only in six, or more, dimensions that microstate geometries have a rich enough moduli space to sample black-hole microstates with sufficient density to obtain a semi-classical thermodynamic model.

One proposal to obtain such solutions has been to place arbitrarily-shaped supertubes in

[1] In this paper we use the terms "BPS" and "supersymmetric" interchangingly, as is common in the literature on supersymmetric supergravity solutions.

[2] See, for example, [12, 13, 14, 15, 16, 17] for reviews.

[3] Despite this, these solutions can sample the typical sector of the underlying conformal field theory [24, 22, 23, 25] and give us insights on how a typical microstate geometry looks.

three-charge bubbling solutions, and thus obtain infinite-dimensional families of three-charge black-hole microstate geometries [26]. Via the so-called "entropy enhancement" mechanism, these supertubes are expected to have more entropy than supertubes in flat space [27, 28]. A rough estimate is that the entropy of normal supertube scales like $\sqrt{Q^2}$, while the entropy of a single enhanced supertube scales like $\sqrt{Q^{5/2}}$, which is a significant increase but still below the black hole scaling, $\sqrt{Q^3}$.

All the supertube entropy counts are done essentially using a Cardy-type formula, in which the central charge is the number of arbitrary continuous functions that determine the supergravity solution. Hence, the obvious way to bypass the $\sqrt{Q^{5/2}}$ bound is to find solutions that depend on a very large number of functions of one variable. One possibility is to consider multiple wiggly supertubes, possibly rotating in opposite directions. In the probe approximation, such tubes have infinite entropy, and there is hope that upon back-reaction this entropy will grow as $\sqrt{Q^3}$ [27]. Unfortunately, the back-reaction calculation is even more complicated than for a single wiggly supertube [26] (which is not particularly simple either) and the fact that one knows the Green's functions for only very specific ambipolar backgrounds makes the extension of this work to general backgrounds all the more difficult.

However, there is now the possibility of a new, even richer candidate microstate geometry: The *superstratum* [29]. Based upon a careful analysis of the supersymmetry structure of three-charge brane configurations, it was argued in [29] that configurations with three electric charges can be given additional dipole charges to create a new class of branes whose shape is given by functions of *two* variables. These configurations are intrinsically six-dimensional, and have 16 supersymmetries locally (in each of the tangent spaces of the superstratum) but globally preserve only the four supersymmetries preserved by their three electric charges.

What distinguishes the superstratum from other exotic branes configurations [30] is that in the D1-D5-P duality frame all the branes that form the superstratum can be given a geometric description and the superstratum supergravity solution is very likely to be *smooth*. Thus the superstratum, if it exists in as full generality as expected, will give rise to microstate geometries that depend upon functions of two variables and these will necessarily provide an even richer sampling of microstates of the black hole. In particular, it is quite plausible that quantizing the fluctuations in one direction will give rise to a parametrically-large number of functions of one variable, which in turn will increase the central charge in the entropy counting, and perhaps give rise to an entropy that grows like $\sqrt{Q^3}$

As explained in [29], a superstratum can become *rigid* when the functions of two variables that determine its shape become functions of only one variable. Such a rigid superstratum is no longer intrinsically six-dimensional but five dimensional, and has eight supersymmetries. It is, in fact, a supertube.

The highly non-trivial aspect of the conjecture in [29] is that a superstratum can be given arbitrary shape modes that are functions of two variables and yet remain smooth. Indeed, the construction of the superstratum was based upon making two successive supertube transitions on the original set of D-branes and the functions of two variables emerge because one can make the two transitions independently so that at each point on the first supertube profile one can choose an independent profile for the second supertube. The regularity of the end result comes about for the same reason that the D1-D5 supertube is smooth [31, 32].

While this conjecture is very plausible it still remains to be proven and one way to achieve this is to construct it explicitly. The general superstratum has D1, D5 and P charges, and lives in type IIB supergravity compactified on T^4 or $K3$. Its solution is therefore a supersymmetric solution of minimal, ungauged, six-dimensional supergravity coupled to one or several anti-self-dual tensor multiplet. Fortunately, the equations governing all these supersymmetric solutions have been written down in [9], generalizing earlier work on the minimal supergravity [7]. Thus we already know what equations the superstratum solution will solve. However, as anybody familiar with [7, 9] knows, finding an intrinsically six-dimensional solution to these complicated equations is highly non-trivial (see for example [33] for one such solution).

Our purpose in this paper is to take a step towards the construction of the superstratum solution by simplifying the non-linear equations of [7, 9] and showing that in fact they can be solved in a linear procedure! The linear structure we uncover here is similar to the one that governs all supersymmetric solutions of *five-dimensional* ungauged supergravity [3]. Indeed, the six-dimensional BPS system also has some remarkable similarities with the generalized linear system, discovered in [34], describing families of non-BPS solutions with floating branes.

The fact that the six-dimensional BPS problem is linear means that the various elements of a complicated superstratum solution will only interact in certain way, and in particular that multiple superstrata can be superposed, and will most likely not destroy each other's wiggles (as might happen in a system governed by non-linear equations), much like multiple wiggly supertubes constructed in five-dimensional supergravity do not hinder each other's oscillations because of the linearity of the five-dimensional solution. Hence, once a superstratum solution is obtained we expect to be able to superpose them and to study the moduli space of multiple superstrata.

We will leave the construction of the general superstratum for future work and begin here by exposing the linear structure of the six-dimensional equations. Then, to illustrate the power of the linear system and to make a first step toward the superstratum, we construct a solution describing multiple interacting supertube spirals with D1, D5 and P charge. These are the charges of a superstratum and by giving them an arbitrary profile we are making the first of the two supertube transitions that lead to the superstratum [29].

Section 2 starts by summarizing the essential results about the six-dimensional supergravity theories [7, 9], and then goes on to re-write the system of equations in a manner that exposes the underlying linearity. Section 3 contains some further simplifications to the system of equations and identifies the electric and magnetic fluxes. Section 4 contains some examples of new solutions that can be generated using this linear structure and Section 5 points to a broad new class of solutions that should be within relatively easy reach for analysis. Section 6 contains our conclusions.

2 Supersymmetric backgrounds

Minimal ungauged $\mathcal{N}=1$ supergravity in six dimensions has a bosonic field content consisting of a graviton and a two-index tensor gauge field, $B_{\mu\nu}$, whose field strength, G, is required to be self-dual. If one reduces this to five dimensions one obtains $\mathcal{N}=2$ supergravity coupled to one vector multiplet. We want to consider the slightly more general, "non-minimal" theory

in six dimensions whose dimensional reduction yields $\mathcal{N}=2$ supergravity coupled to two vector multiplets. The corresponding $\mathcal{N}=1$ supergravity in six dimensions is simply the minimal theory coupled to an extra anti-self-dual tensor multiplet. The complete bosonic field content is then a graviton, a *general* two-index tensor gauge field, $B_{\mu\nu}$, and a dilaton, ϕ.

The conditions for the most general supersymmetric solutions of the minimal $\mathcal{N}=1$ gauged supergravity in six dimensions were first obtained in [7]. In [9], this was then generalized to an even larger class of six-dimensional $\mathcal{N}=1$ supergravity theories coupled to vector multiplets. One can then extract the supersymmetry conditions for the non-minimal ungauged $\mathcal{N}=1$ theory of interest here by setting all the vector fields and gauge couplings to zero in [9].

The first part of the supersymmetry analysis implies that the solution must have a null Killing vector and that all fields must have vanishing Lie derivative along that direction. We will choose a coordinate, u, along this Killing vector and then all fields will be u-independent. However, the fields will, in general, depend upon the other five coordinates. The null Killing vector introduces a $2+4$ split in the geometry and so it is natural to introduce a second retarded time coordinate[4], v, and a four-dimensional, and generically v-dependent, spatial base, $\mathcal{B}$, with coordinates x^m, $m=1,\ldots,4$.

2.1 The metric

The six-dimensional metric is:

$$ds^2 = 2H^{-1}(dv+\beta)\big(du+\omega \;+\; \tfrac{1}{2}\,\mathcal{F}\,(dv+\beta)\big) \;-\; H\,ds_4^2\,, \tag{2.1}$$

where the metric on the four-dimensional base, $\mathcal{B}$, is written in terms of components as:

$$ds_4^2 \;=\; h_{mn}dx^m dx^n\,. \tag{2.2}$$

In (2.1), $\beta=\beta_m dx^m$ and $\omega=\omega_m dx^m$ are regarded as 1-forms on $\mathcal{B}$. The functions H and $\mathcal{F}$, the 1-forms β and ω, and the metric h_{mn} all depend on v and x^m but not on u.

The supersymmetry conditions imply that the base is almost hyper-Kähler in that there are three anti-self-dual 2-forms,

$$J^{(A)} \;\equiv\; \tfrac{1}{2}\,J^{(A)}{}_{mn}\,dx^m\wedge dx^n\,, \tag{2.3}$$

that satisfy the algebra:

$$J^{(A)m}{}_pJ^{(B)p}{}_n = \epsilon^{ABC}\,J^{(C)m}{}_n \;-\; \delta^{AB}\,\delta^m_n\,. \tag{2.4}$$

These forms are also required to satisfy the differential identity:

$$\tilde{d}J^{(A)} \;=\; \partial_v\big(\beta\wedge J^{(A)}\big)\,, \tag{2.5}$$

where $\tilde{d}$ is the exterior derivative restricted to $\mathcal{B}$, which acts on a p-form, $\Phi\in\Lambda^p(\mathcal{B})$, by:

$$\Phi \;=\; \frac{1}{p!}\,\Phi_{m_1\ldots m_p}(x,v)\,dx^{m_1}\wedge\ldots\wedge dx^{m_p}\,, \tag{2.6}$$

$$\tilde{d}\Phi \;\equiv\; \frac{1}{(p+1)!}\,(p+1)\,\frac{\partial}{\partial x^{[q}}\Phi_{m_1\ldots m_p]}\,dx^q\wedge dx^{m_1}\wedge\ldots\wedge dx^{m_p}\,. \tag{2.7}$$

[4]Note that we are reversing the conventions of [7, 9], by interchanging the roles of the coordinates u and v.

Also, $\partial_v \Phi$ denotes the Lie derivative of a quantity Φ with respect to the tangent vector $\frac{\partial}{\partial v}$. We will also use the notation $\dot{\Phi} \equiv \partial_v \Phi$. Note that we are using the conventions of [7] in the definition of the $J^{(A)}$ and that one must make the replacement $J^{(A)} \to -J^{(A)}$ in order to go to the conventions of [9].

There is a remark after Equation (4.13) in [9] that outlines a simple computation that purportedly leads to another differential constraint on the $J^{(A)}$. However one can use the explicit spin connections given in [7] to perform this computation and we find, exactly as noted in [7], that this computation simply yields (2.5) and there are no other constraints on $J^{(A)}$.

Define the anti-self-dual 2-forms, ψ and $\hat{\psi}$, by:

$$\psi \;\equiv\; H \hat{\psi} \;\equiv\; \tfrac{1}{16} H \,\epsilon^{ABC} \, J^{(A)mn} \dot{J}^{(B)}{}_{mn} \, J^{(C)} \,, \tag{2.8}$$

Note that $\hat{\psi}$ is anti-self-dual and depends solely upon the almost hyper-Kähler structure on the base. It should be noted that both references [7] and [9] use the same definition of ψ and so, given our choice in (2.4), we have the same sign conventions for ψ as in [7] and that one should send $\psi \to -\psi$ to go to the conventions of [9].

Following [7], we introduce the frames, $\{e^+, e^-, e^a\}$ in which the metric takes the form:

$$ds^2 \;=\; 2e^+ e^- \;-\; \delta_{ab}\, e^a \, e^b \,; \tag{2.9}$$

$$e^+ \equiv H^{-1}\big(dv + \beta\big) \,, \qquad e^- \;\equiv\; du + \omega + \tfrac{1}{2}\, \mathcal{F} H \, e^+ \,, \qquad e^a = H^{\frac{1}{2}} \tilde{e}^a{}_m dx^m \,, \tag{2.10}$$

and we use the orientation

$$\epsilon^{+-1234} \;=\; \epsilon^{1234} \;=\; +1 \,. \tag{2.11}$$

In d dimensions, we define the Hodge star $*_d$ to act on a p-form as

$$*_d \left(dx^{m_1} \wedge \cdots \wedge dx^{m_p}\right) = \frac{1}{(d-p)!} dx^{n_1} \wedge \cdots \wedge dx^{n_{d-p}} \, \epsilon_{n_1 \ldots n_{d-p}}{}^{m_1 \ldots m_p} \,. \tag{2.12}$$

It is convenient to introduce the Kaluza-Klein covariant differential operator, D, defined by:

$$D\Phi \;\equiv\; \tilde{d}\Phi \;-\; \beta \wedge \dot{\Phi}. \tag{2.13}$$

The vector field, β, is then required to satisfy the self-duality condition:

$$D\beta \;=\; *_4 \, D\beta \,. \tag{2.14}$$

Note that if one acts with $\tilde{d}$ on both sides of (2.5), and then re-uses the equation, one obtains the integrability condition: $\partial_v (D\beta \wedge J^{(A)}) = 0$, which follows from the self-duality of $D\beta$ and the anti-self-duality of $J^{(A)}$.

The first step in obtaining a supersymmetric background is to select an almost hyper-Kähler base, whose almost complex structures satisfy (2.4) and (2.5), and then obtain a vector field, β, satisfying (2.14). We will show below that this represents the only non-linear aspect of finding the most general supersymmetric background and that all subsequent steps can be reduced to a completely linear system of equations.

We note that one can, of course, simplify things by taking the base to be hyper-Kähler and restricting β to be independent of v. In particular, this means that $\psi = 0$ and that the equation (2.14) is a simple, linear self-duality condition on the base.

2.2 The tensor gauge field

Supersymmetry requires that the self-dual parts (in six dimensions) of the field strength, G, and spin connection be the same. This means that the supersymmetries will be constant in this coordinate system and frame:

$$\partial_\mu \epsilon \;=\; 0\,. \tag{2.15}$$

The remaining components of the field strength can be written in terms of the dilaton and 2-from, K, that is self-dual on the four-dimensional base. The final expression may be found in [9]:

$$\begin{aligned} e^{\sqrt{2}\phi}\, G \;=\;& \tfrac{1}{2}\, *_4 \big(DH + H\,\dot\beta - \sqrt{2}\,H\,D\phi\big) \\ & -\tfrac{1}{2}\, e^+ \wedge e^- \wedge \big(H^{-1}DH + \dot\beta + \sqrt{2}\,D\phi\big) \\ & -e^+ \wedge \big(-H\psi + \tfrac{1}{2}\,(D\omega)^- - K\big) + \tfrac{1}{2}\,H^{-1}\,e^- \wedge D\beta\,, \end{aligned} \tag{2.16}$$

and, for completeness, we give:

$$\begin{aligned} e^{\sqrt{2}\phi}\; *_6 G \;=\;& \tfrac{1}{2}\, *_4 \big(DH + H\,\dot\beta + \sqrt{2}\,H\,D\phi\big) \\ & -\tfrac{1}{2}\, e^+ \wedge e^- \wedge \big(H^{-1}DH + \dot\beta - \sqrt{2}\,D\phi\big) \\ & -e^+ \wedge \big(-H\psi + \tfrac{1}{2}\,(D\omega)^- + K\big) + \tfrac{1}{2}\,H^{-1}\,e^- \wedge D\beta\,. \end{aligned} \tag{2.17}$$

where we define:

$$(D\omega)^\pm \;\equiv\; \tfrac{1}{2}\,(D\omega \pm *_4 D\omega)\,. \tag{2.18}$$

Our expressions for (2.16) and (2.17) differ from those of [9] by replacing $\psi \to -\psi$ as a result of the difference in conventions outlined above.

The Bianchi identity and the equation of motion are simply:

$$d\,G \;=\; 0\,, \qquad d\big(e^{2\sqrt{2}\phi}\; *_6 G\big) \;=\; 0\,, \tag{2.19}$$

and they imply the following differential identities[5]:

$$\begin{aligned} D\big(H^{-1}e^{\sqrt{2}\phi}(K - H\mathcal{G} - H\psi)\big) \;+\;& \tfrac{1}{2}\,\partial_v *_4 \big(D(H\,e^{\sqrt{2}\phi}) + H\,e^{\sqrt{2}\phi}\dot\beta\big) \\ -\;& H^{-1}e^{\sqrt{2}\phi}\,\dot\beta \wedge (K - H\mathcal{G} - H\psi) \;=\; 0\,, \end{aligned} \tag{2.20}$$

$$\begin{aligned} -D\big(H^{-1}e^{-\sqrt{2}\phi}(K + H\mathcal{G} + H\psi)\big) \;+\;& \tfrac{1}{2}\,\partial_v *_4 \big(D(H\,e^{-\sqrt{2}\phi}) + H\,e^{-\sqrt{2}\phi}\dot\beta\big) \\ +\;& H^{-1}e^{-\sqrt{2}\phi}\,\dot\beta \wedge (K + H\mathcal{G} + H\psi) \;=\; 0\,, \end{aligned} \tag{2.21}$$

and

$$D *_4 \big[D(H\,e^{\sqrt{2}\phi}) + H\,e^{\sqrt{2}\phi}\dot\beta\big] \;=\; 2\,H^{-1}e^{\sqrt{2}\phi}(K - H\mathcal{G}) \wedge D\beta\,, \tag{2.22}$$

$$D *_4 \big[D(H\,e^{-\sqrt{2}\phi}) + H\,e^{-\sqrt{2}\phi}\dot\beta\big] \;=\; -2\,H^{-1}e^{-\sqrt{2}\phi}(K + H\mathcal{G}) \wedge D\beta\,. \tag{2.23}$$

where $\mathcal{G}$ is defined as in [9][6]:

$$\mathcal{G} \;\equiv\; \frac{1}{2H}\big[\,(D\omega)^+ \;+\; \tfrac{1}{2}\,\mathcal{F}\,D\beta\,\big]\,. \tag{2.24}$$

Note that this form is self-dual in four dimensions.

[5] We have also corrected sign errors in equation (4.19) of [9] that would otherwise render it inconsistent with [7] once one has replaced $\psi \to -\psi$.

[6] This differs by a factor of two compared to the definition in [7].

2.3 The linear structure

To expose the linear structure, one first defines the following forms and functions

$$\Theta_1 \;\equiv\; H^{-1}e^{-\sqrt{2}\phi}(K + H\mathcal{G} + H\psi)\,, \qquad \Theta_2 \;\equiv\; H^{-1}e^{\sqrt{2}\phi}(-K + H\mathcal{G} + H\psi)\,, \tag{2.25}$$

$$Z_1 \;\equiv\; H\,e^{\sqrt{2}\phi}\,, \qquad Z_2 \;\equiv\; H\,e^{-\sqrt{2}\phi}\,. \tag{2.26}$$

Note that the Θ_j, $j = 1, 2$, are almost self-dual:

$$*_4\,\Theta_1 \;=\; \Theta_1 - 2\,e^{-\sqrt{2}\phi}\,\psi \;=\; \Theta_1 - 2\,Z_2\,\hat{\psi}\,, \tag{2.27}$$

$$*_4\Theta_2 \;=\; \Theta_2 - 2\,e^{\sqrt{2}\phi}\,\psi \;=\; \Theta_2 - 2\,Z_1\,\hat{\psi}\,. \tag{2.28}$$

In particular, the anti-self-dual parts of the Θ_j are proportional to $\hat{\psi}$.

With these definitions, equations (2.20) and (2.22) become

$$\tilde{d}\Theta_2 \;=\; \partial_v\big[\tfrac{1}{2}\,*_4\,(DZ_1 + \dot{\beta}Z_1) \;+\; \beta\wedge\Theta_2\big]\,, \tag{2.29}$$

$$D\,*_4\,(DZ_1 + \dot{\beta}Z_1) = -2\,\Theta_2\wedge D\beta\,, \tag{2.30}$$

while (2.21) and (2.23) become

$$\tilde{d}\Theta_1 \;=\; \partial_v\big[\tfrac{1}{2}\,*_4\,(DZ_2 + \dot{\beta}Z_2) \;+\; \beta\wedge\Theta_1\big]\,, \tag{2.31}$$

$$D\,*_4\,(DZ_2 + \dot{\beta}Z_2) = -2\,\Theta_1\wedge D\beta\,. \tag{2.32}$$

Note that these equations and the duality properties of the Θ_j also imply:

$$\tilde{d}(*_4\Theta_1) \;=\; -\,2\,\tilde{d}(Z_2\hat{\psi}) + \partial_v\big[\tfrac{1}{2}\,*_4\,(DZ_2 + \dot{\beta}Z_2) \;+\; \beta\wedge\Theta_1\big]\,, \tag{2.33}$$

and

$$\tilde{d}(*_4\Theta_2) \;=\; -\,2\,\tilde{d}(Z_1\hat{\psi}) + \partial_v\big[\tfrac{1}{2}\,*_4\,(DZ_1 + \dot{\beta}Z_1) \;+\; \beta\wedge\Theta_2\big]\,, \tag{2.34}$$

Since β and $\hat{\psi}$ are known fields on the base manifold, equations (2.29), (2.30) and (2.34) define a *linear system* for Θ_2 and Z_1, while equations (2.31), (2.32) and (2.33) define a *linear system* for Θ_1 and Z_2.

It follows that once we have fixed the base manifold, $\mathcal{B}$, and the vector field β, the Z_j and Θ_j are determined by linear equations. One can then invert (2.26) and (2.25) to determine H, ϕ, K and $\mathcal{G}$.

2.4 The angular momentum vector and the last metric function

The last two equations that determine the supersymmetric background come from inverting (2.24):

$$(D\omega)^+ \;=\; 2\,H\,\mathcal{G} \;-\; \tfrac{1}{2}\mathcal{F}\,D\beta\,. \tag{2.35}$$

and from the one Einstein equation that is not implied by the supersymmetry conditions.

To write the latter equation it is convenient to define:

$$L \;\equiv\; \dot{\omega} \;+\; \tfrac{1}{2}\,\mathcal{F}\dot{\beta} \;-\; \tfrac{1}{2}\,D\mathcal{F}\,. \tag{2.36}$$

Note that this is gauge invariant under the transformation:

$$\mathcal{F} \;\to\; \mathcal{F} + 2\,\partial_v f\,, \qquad \omega \;\to\; \omega + Df\,, \tag{2.37}$$

for some function, $f(v, x^m)$. This transformation is induced by a coordinate change $u \to u + f(v, x^m)$ in the metric (2.1). The last necessary equation may then be written [9]:

$$\begin{aligned} *_4\, D *_4 L \;&=\; \tfrac{1}{2}\, H h^{mn}\partial_v^2(H h_{mn}) + \tfrac{1}{4}\,\partial_v(H h^{mn})\,\partial_v(H h_{mn}) - 2\,\dot{\beta}_m\, L^m + 2\, H^2\, \dot{\phi}^2 \\ &\quad -\tfrac{1}{2}\, H^{-2}\big(D\omega + \tfrac{1}{2}\,\mathcal{F} D\beta\big)^2 + 2\, H^{-2}\big(K - H\,\psi + \tfrac{1}{2}(D\omega)^-\big)^2\,. \end{aligned} \tag{2.38}$$

where, for any 2-form, $\mathcal{M}^2 = \frac{1}{2} M_{mn} M^{mn}$.

One can now rewrite this apparently non-linear equation in ω by first replacing $D\omega$ by $(D\omega)^- + (D\omega)^+$ and expanding the squares. The $((D\omega)^-)^2$ terms cancel and one can then replace $(D\omega)^+$ using (2.35) and the definitions of the Θ_j's. Further simplifications can then be made by using the duality properties of all 2-forms and the fact that the contraction of a self-dual and anti-self dual form vanishes identically. We then find that (2.38) can be rewritten as:

$$\begin{aligned} *_4\, D *_4 L \;&=\; \tfrac{1}{2}\, H h^{mn}\partial_v^2(H h_{mn}) + \tfrac{1}{4}\,\partial_v(H h^{mn})\,\partial_v(H h_{mn}) - 2\,\dot{\beta}_m\, L^m + 2\, H^2\, \dot{\phi}^2 \\ &\quad -2 *_4 \Big[\, \Theta_1 \wedge \Theta_2 \;-\; H^{-1}\psi \wedge D\omega \Big]\,. \end{aligned} \tag{2.39}$$

In re-writing the right-hand side of (2.39) in terms of the dual of wedge products of two forms, rather than the contractions of two forms, it is important to remember that the latter involves the *sum* of the contractions of the self-dual and anti-self-dual parts, whereas former produces the *difference* between the contractions of the self-dual and the anti-self-dual parts. Thus one needs to employ the duality properties (2.27) and (2.28) and the the anti-self-duality of ψ to arrive at the correct expression for (2.39).

Finally, (2.35) can be rewritten as:

$$\begin{aligned} D\omega + *_4 D\omega \;&=\; 2\, Z_1\, \Theta_1 + 2\, Z_2\, \Theta_2 \;-\; \mathcal{F}\, D\beta - 4\, H\, \psi \\ &=\; 2\, Z_1 \big(\Theta_1 - Z_2\, \hat{\psi}\big) + 2\, Z_2 \big(\Theta_2 - Z_1\, \hat{\psi}\big) \;-\; \mathcal{F}\, D\beta\,. \end{aligned} \tag{2.40}$$

In the second identity we have arranged the right hand side into manifestly self-dual combinations that follow from (2.27) and (2.28).

Note that (2.39) and (2.40) are *linear* in $\mathcal{F}$ and ω. Since h_{mn}, H, ϕ, Θ_1, Θ_2, β and ψ are determined by the base and by solving the earlier linear systems, we see that (2.39) and (2.40) define another *linear system* that can be used to determine $\mathcal{F}$ and ω. Thus, once one has fixed the base geometry and one has determined β, the entire solution is determined by a linear structure. In particular, vast numbers of new solutions can be obtained by superposition.

3 Simplifying the fluxes and flux equations

Define the 3-forms

$$\gamma_1 \;\equiv\; \tfrac{1}{2} *_4 (DZ_1 + \dot\beta Z_1) \;+\; \beta\wedge\Theta_2\,, \qquad \gamma_2 \;\equiv\; \tfrac{1}{2} *_4 (DZ_2 + \dot\beta Z_2) \;+\; \beta\wedge\Theta_1\,. \tag{3.1}$$

Then equations (2.29), (2.30), (2.31) and (2.32) may be written very simply as:

$$\tilde d\Theta_2 \;=\; \partial_v\gamma_1\,, \qquad \tilde d\gamma_1 \;=\; 0\,; \qquad \tilde d\Theta_1 \;=\; \partial_v\gamma_2\,, \qquad \tilde d\gamma_2 \;=\; 0\,. \tag{3.2}$$

The tensor gauge field (2.16) and its dual (2.17) can be written in terms of electric and magnetic parts:

$$G \;=\; d\big[-\tfrac{1}{2}\,Z_1^{-1}\,(du+\omega)\wedge(dv+\beta)\big] \;+\; \widehat{G}_1\,, \tag{3.3}$$

$$e^{2\sqrt{2}\phi} *_6 G \;=\; d\big[-\tfrac{1}{2}\,Z_2^{-1}\,(du+\omega)\wedge(dv+\beta)\big] \;+\; \widehat{G}_2\,, \tag{3.4}$$

where

$$\widehat{G}_1 \;\equiv\; \tfrac{1}{2} *_4 (DZ_2 + \dot\beta Z_2) \;+\; (dv+\beta)\wedge\Theta_1\,, \tag{3.5}$$

$$\widehat{G}_2 \;\equiv\; \tfrac{1}{2} *_4 (DZ_1 + \dot\beta Z_1) \;+\; (dv+\beta)\wedge\Theta_2\,. \tag{3.6}$$

In particular, the Bianchi identities and Maxwell equations then require the closure of the $\widehat{G}_j$, which means these quantities do indeed measure the conserved magnetic charge.

Computing $d\widehat{G}_j$ one easily finds:

$$D\big[*_4 (DZ_i + \dot\beta Z_i)\big] \;+\; 2\,(D\beta)\wedge\Theta_j \;=\; 0\,, \tag{3.7}$$

$$D\Theta_j - \dot\beta\wedge\Theta_j \;-\; \partial_v\big[\tfrac{1}{2} *_4 (DZ_i + \dot\beta Z_i)\big] \;=\; 0\,. \tag{3.8}$$

where $\{i,j\} = \{1,2\}$. One can easily check that these equations are equivalent to (3.2).

The closure of $\widehat{G}_j$ also means that they are locally exact: $\widehat{G}_j = d\xi_j$. Moreover, because all the fields are independent of u and there are no du terms in $\widehat{G}_j$, one can assume that the 2-forms, ξ_j, also do not involve any du terms. One can then make a gauge choice for the ξ_j so as to remove all the dv terms[7]. This means that one can write:

$$\tfrac{1}{2} *_4 (DZ_i + \dot\beta Z_i) \;=\; D\xi_j\,, \qquad \Theta_j \;=\; \partial_v\,\xi_j\,. \tag{3.9}$$

The integrability conditions for ξ_j are, of course, the equations (3.7) and (3.8).

Now recall the duality properties, (2.27) and (2.28), of the Θ_j. These imply that the Θ_j can be written entirely in terms of three arbitrary functions on the base and if one expands in Fourier modes, one can use the second equation in (3.9) to write the ξ_j in terms of three arbitrary functions. The first equation in (3.9) is, in fact, four independent equations, one for each component, and so these equations can, in principle, be used to determine the four unknown functions: Z_i and the three functions in ξ_j.

[7] Strictly speaking one can only do this for the non-trivial Fourier modes along the v-direction. The computation for the zero-mode will, of course, reproduce the usual five-dimensional BPS equations.

4 Examples

In this section, we use our linear procedure to construct several new solutions. These solutions can be thought of as configurations of D1 and D5 branes in type IIB string theory compactified to six dimensions. Indeed, the unconstrained two-index gauge field $B_{\mu\nu}$ of the six-dimensional theory corresponds to the RR 2-form, $C_{\mu\nu}$, and this is sourced by D1-branes and D5-branes. The scalar, ϕ, corresponds to $\frac{\Phi}{\sqrt{2}}$ where Φ is the ten-dimensional dilaton.

4.1 The D1-P supertube spiral

The first solution corresponds to the bound state of D1 branes along the v direction and momentum (P) along the v direction. This D1-P system is dual to the well-known F1-P system [35] and the bound state is described by the D1 world-volume moving along $\vec{x} = \vec{F}(v)$ where $\vec{x} = (x^1, \ldots, x^4)$ and $\vec{F}(v)$ is an arbitrary vector function. This solution is not new, but casting it in our formalism is instructive for the construction of the more general solutions described in the next subsection.

The metric and RR 2-form can be obtained by dualizing the known supergravity solution of the F1-P system [36, 37] and putting it into our form. The resulting fields are:

$$\begin{aligned} h_{mn} &= \delta_{mn}, \qquad \beta = \psi = 0, \\ Z_1 &= 1 + H_1, \qquad H_1 = \frac{Q_1}{|\vec{x} - \vec{F}(v)|^2}, \qquad Z_2 = 1, \\ \Theta_1 &= 0, \qquad \Theta_2 = \frac{1}{2}(1 + *_4)\, \tilde{d}(H_1 \dot{F}_m dx^m), \\ \omega &= H_1 \dot{F}_m dx^m, \qquad \mathcal{F} = -H_1 \dot{F}^2, \qquad K = -\frac{1}{2}\Theta_2. \end{aligned} \tag{4.1}$$

The base space of this solution is flat $\mathbb{R}^4$. The one-form, β, is sourced by KK monopole charges whose special direction is v and, since the solution has no such charge, β vanishes. Hence, the v fiber is trivial and the spatial metric is a warped of $\mathbb{R}^4 \times S^1$. Because the D1 world-volume is not straight but along the curve $\vec{x} = \vec{F}(v)$, there are D1 charges along v as well as $\vec{x}$, and they are respectively encoded in Z_1 and Θ_2. Also, because the D1 world-volume is moving, there are momentum charges along v and $\vec{x}$, and they are respectively encoded in $\mathcal{F}$ and ω.

The foregoing solution involves only one "strand" of D1-branes, but one can easily include multiple strands by summing over strands:

$$Z_1 = 1 + \sum_{p=1}^{n} \frac{Q_{1p}}{|\vec{x} - \vec{F}_p(v)|^2}. \tag{4.2}$$

Each function, $\vec{F}_p(v)$, is arbitrary and describes the position of the p^{th} strand. The other fields, Θ_2, ω and $\mathcal{F}$, are given by expressions similar to (4.1) but now including summation over multiple strands.

4.2 D1-D5-P supertube

It is relatively simple to add D5-branes extending along the v direction[8] and obtain a more general traveling-wave solution. This solution will have D1, D5 and P charges, and describes a traveling momentum wave on a D1-D5 system. The resulting arbitrarily-shaped profile is a three-charge two-dipole charge supertube spiral, and our solution describes an arbitrary distribution of such parallel spirals. To our knowledge this solution is new. As discussed in [29], this configuration corresponds to the first of the two supertube transitions involved in the "double bubbling:" the D1-D5-P system is polarized into a three-charge supertube spiral of arbitrary shape.

Just as in the D1-P example, we take the base to be flat and, because this tube will have no KKM dipole moment, we again set β to zero[9]. Hence

$$h_{mn} = \delta_{mn}, \qquad \beta = \psi = 0\,, \tag{4.3}$$

which implies that the Z_i equations and Θ_i equations decouple. The equations (2.30), (2.32) simply give the Laplace equation on the base:

$$\tilde{d} *_4 \tilde{d}\, Z_i = 0, \qquad i = 1, 2. \tag{4.4}$$

Following the D1-P example, we choose the following harmonic functions:

$$Z_i = c_i + \sum_{p=1}^{n} \frac{Q_{ip}}{|\vec{x} - \vec{F}(v) - \vec{a}_p|^2} \tag{4.5}$$

with constants, c_i, an arbitrary vector function, $\vec{F}(v)$, and *constant* vectors, $\vec{a}_p$. Because singularities of Z_1 and Z_2 represent the D1 and D5 charges, respectively, this solution corresponds to n *parallel* strands of D1-D5 world-volume along $\vec{x} = \vec{F}(v) + \vec{a}_p$. For later convenience, we define:

$$H_i = \sum_{p=1}^{n} \frac{Q_{ip}}{|\vec{x} - \vec{F}(v) - \vec{a}_p|^2}\,, \qquad i = 1, 2\,. \tag{4.6}$$

The magnetic components, Θ_i, of the solution are determined by (2.29) and (2.31), which now become:

$$\tilde{d}\Theta_i = \frac{1}{2} *_4 \tilde{d}\dot{Z}_j, \qquad \{i, j\} = \{1, 2\}. \tag{4.7}$$

Their solution is simply:

$$\Theta_i = \frac{1}{2}(1 + *_4)\, \tilde{d}(H_j \dot{F}_m dx^m)\,. \tag{4.8}$$

[8]The D5-branes also wrap the four internal directions of the compactification of the IIB theory to six dimensions.

[9]Adding the KKM dipole moment along a second, independent profile represents the second supertube transition of the "double bubbling" and would greatly increase the technical challenge of finding the solution.

One also has the freedom to add closed forms to Θ_i. The functions H, ϕ, K are given by

$$H = Z_1^{1/2} Z_2^{1/2}, \qquad e^{2\sqrt{2}\phi} = \frac{Z_1}{Z_2}, \qquad K = \frac{1}{2}(Z_1\Theta_1 - Z_2\Theta_2), \tag{4.9}$$

Since the Θ_i are not closed, they do not define conserved magnetic charges. Indeed, as noted in Section 3, the conserved magnetic charges of six-dimensional solutions are defined by integrating the 3-forms, $\widehat{G}_i$, defined in (3.5) and (3.6), over three cycles. The only non-trivial 3-cycles in our D1-D5 strands are the Gaussian S^3's that surround each strand in the five-dimensional spatial metric. The corresponding flux integrals then give the only quantized charges in the solution: the Q_{jp}. On the other hand, one can imagine smearing the solution along the v fiber and compactifying along that direction. One now has a profile along the base and a Gaussian S^2 surrounding that profile. This means that there is a non-trivial 3-cycle defined by this S^2 and the v fiber. The integral of $\widehat{G}_i$ over this cycle reduces to the integral of Θ_i over the S^2 and, from (4.8), the magnetic charges of the smeared profile defined in this way will be proportional to $Q_{ip}|\dot{F}|$. Thus the smeared versions of our new solutions[10] will have electric charges, Q_{ip}, and magnetic dipole charges given by $Q_{ip}|\dot{F}|$.

With the simple background defined by (4.3), the ω and L equations are decoupled and the ω equation (2.40) becomes

$$(1 + *_4)\tilde{d}\omega \;=\; 2(Z_1\Theta_1 + Z_2\Theta_2). \tag{4.10}$$

It is easy to check that the following is a solution:

$$\omega \;=\; (H_1H_2 + c_1H_2 + c_2H_1)\dot{F}_m dx^m. \tag{4.11}$$

This has a pole structure familiar from the 3-charge 2-dipole-charge black rings, but now everything depends on v as well. Again, in (4.11), we could have added a closed form to ω.

Finally, $\mathcal{F}$ is determined by (2.39), which can be written as

$$\begin{aligned} *_4\tilde{d} *_4 \tilde{d}\,\mathcal{F} = -\partial_m\partial_m\mathcal{F} &= 2 *_4 \tilde{d} *_4 \dot{\omega} - 2(\dot{Z}_1\dot{Z}_2 + Z_1\ddot{Z}_2 + \ddot{Z}_1 Z_2) + 4 *_4 (\Theta_1 \wedge \Theta_2) \\ &= 2\,\partial_m H_1\,\partial_m H_2\,\dot{F}^2. \end{aligned} \tag{4.12}$$

In deriving this expression and some of the expressions above, the relation $\dot{Z}_i = \dot{H}_i = -\dot{F}_m\,\partial_m Z_i = -\dot{F}_m\,\partial_m H_i$ is useful. Because H_i are harmonic with respect to $\vec{x}$, we find

$$\mathcal{F} \;=\; -\,H_1H_2\,\dot{F}^2 \;+\; H_3\,, \tag{4.13}$$

where we have chosen to add explicitly a function, $H_3(v,\vec{x})$, that is harmonic on the four-dimensional base. This function corresponds to putting a freely-choosable momentum profile on the D1-D5 world-volume.

When we only have one strand ($n = 1$), with equal D1 and D5 charges and harmonic functions ($c_1 = c_2 = 1$, $Q_1 = Q_2$), then our solution reduces to the black string with traveling waves of [38, 39]. The closed form that we could have added in (4.11) corresponds to the angular

[10] While we are not explicitly smearing the solution here, the second supertube transition in double bubbling does smear the solution and these considerations will become important in future work.

momentum distribution discussed in [39] while the harmonic form H_3 in (4.13) corresponds to the momentum distribution studied in [38]. Our general solution describes a parallel distribution of black strings whose traveling waves have identical profiles. For a generic momentum distribution the strings will have a nonzero horizon area, and when the source of H_3 is such that the classical horizon area is zero, the black strings with traveling waves become three-charge two-dipole-charge supertube strands.

It is interesting to note that the solution for two-charge supertube spirals presented in the previous subsection and the solution for multiple D1-P and D5-P strands and black strings with traveling waves given here both have a certain "action at a distance" property: if one slices the solution at a constant value of v, then the fields on that slice only depend on the positions of the strands and on their v-derivatives on that slice, but are completely independent of the behavior of the strands at any other values of v! Hence, if one thinks of the supertube strands as traveling waves on D1 and D5 branes, the fields at some location (x_i, v_0) far away from the strand are independent of the values of the functions at any point except v_0.

This "action at a distance" behavior gives some insight into why the solution with parallel strands is much more straightforward to find than solutions with different profiles for each strand. If one takes constant-v slices of a solution in which all the profiles are the same then, in any slice, the distance between the strands and the derivatives of the F_i are the same. Since these derivatives give the dipole charges, it is not hard to see that this implies that, in any particular slice, the symplectic product between the charges of any two centers is zero. Hence, there are no bubble equations, or integrability conditions, to govern the separation between parallel strands and these strands can be moved at any position, as long as they remain parallel.

To illustrate this, it is easier to consider a D1-P strand and a parallel D5-P strand. In a given slice, the first strand has a D1 dipole charge of magnitude and orientation:

$$d_1 \;=\; Q_1 \dot{\vec{F}}(v) \tag{4.14}$$

and the second strand has a D5 dipole charge:

$$d_5 \;=\; Q_5 \dot{\vec{F}}(v)\,. \tag{4.15}$$

Hence, the symplectic product between the two strands, $Q_1 d_5 - Q_5 d_1$, vanishes exactly in any slice. Given that D1-D5-P three-charge spirals can always be decomposed into parallel D1-P and D5-P strands [29], and given that D1-P strands do not interact with each other (as explained in the previous section) the foregoing calculation indicates that parallel strands can be moved anywhere in the solution and there are no constraints on their position. One can make a similar argument for parallel black strings with identical traveling waves: they only differ from the three-charge supertube strands in the extra "momentum" harmonic function H_3, and since the solution has no KKM charges this extra harmonic function does not affect the bubble equations. The absence of bubble equations can also be understood from the absence of Dirac-Misner strings in the expression, (4.11), for ω.

If we do not assume that different strands are parallel then the linear procedure still gives straightforward solutions for the electric potentials, or warp factors: one simply replaces $\vec{F}(v)$ in (4.5) by $\vec{F}_p(v)$. However, the solution for ω and $\mathcal{F}$ will be significantly more complicated.

Moreover, the dipole charges at various points on each strand will not be parallel to each other, and there will be non-trivial v-dependent bubble equations. That is, requiring the absence of Dirac strings and the absence of closed time-like curves will impose conditions on the locations of the strands on every v-slice. Based on a rough counting, we expect the solution for two strands to be given by seven arbitrary functions: four shapes modes for each strand minus one for the v-dependent bubble equation.

5 An Intermediate Class of Solutions

We have found that the system of equations describing supersymmetric solutions of six-dimensional, minimal, ungauged supergravity coupled to an anti-symmetric tensor multiplet can be solved in a linear process once the base metric and the vector field, β, have been determined. Since the vector field, β, is now a *geometric* field, it may be viewed as defining the background spatial geometry and so the only non-linearities lie in determining this background geometry while the remaining parts of the solution are entirely linear.

When the solutions to this theory do not depend on the "common D1-D5" direction, v, they can be reduced to solutions of yields $\mathcal{N} = 2$ ungauged five-dimensional supergravity coupled to two vector multiplets, and the complicated linear system found here collapses to the simpler linear system found in [3].

There is, however, a very interesting intermediate class of solutions in which β, the complex structures and the base metric are v-independent. The condition (2.5) then requires the complex structures to be closed and thus the base metric must, in fact, be hyper-Kähler. One also has $\hat{\psi} \equiv 0$. The system of equations then collapses to:

$$\tilde{d}\beta \;=\; *_4\, \tilde{d}\beta\,, \qquad *_4\Theta_1 \;=\; \Theta_1\,, \qquad *_4\Theta_2 \;=\; \Theta_2\,, \tag{5.1}$$

$$\tilde{d}\Theta_2 \;-\; \beta\wedge\dot{\Theta}_2 \;=\; \tfrac{1}{2}\, *_4\, D\dot{Z}_1\,, \qquad D *_4 DZ_1 \;=\; -\,2\,\Theta_2\wedge\tilde{d}\beta\,, \tag{5.2}$$

$$\tilde{d}\Theta_1 \;-\; \beta\wedge\dot{\Theta}_1 \;=\; \tfrac{1}{2}\, *_4\, D\dot{Z}_2\,, \qquad D *_4 DZ_2 \;=\; -\,2\,\Theta_1\wedge\tilde{d}\beta\,, \tag{5.3}$$

$$*_4\, D *_4 (\dot{\omega} - \tfrac{1}{2}\, D\mathcal{F}) \;=\; -\,2 *_4 \Theta_1\wedge\Theta_2 \;+\; Z_1\partial_v^2 Z_2 \;+\; Z_2\partial_v^2 Z_1 + (\partial_v Z_1)(\partial_v Z_2)\,, \tag{5.4}$$

$$D\omega + *_4 D\omega \;=\; 2\, Z_1\,\Theta_1 + 2\, Z_2\,\Theta_2 \;-\; \mathcal{F}\,\tilde{d}\beta\,. \tag{5.5}$$

This linear system is similar to that of [3], except that all the functions and fields, with the exception of β, are now allowed to be functions of the fiber coordinate, v. We believe that even with the simplifying assumptions above there are going to be rich new varieties of microstate geometries in six dimensions.

6 Conclusions

Six-dimensional, minimal ungauged supergravity coupled to an anti-symmetric tensor multiplet, when reduced to five dimensions, yields $\mathcal{N} = 2$ supergravity coupled to two vector multiplets. Thus the six-dimensional theory is extremely important to the general study of microstate geometries: smooth geometries that have the same asymptotic structure at infinity as a given

five-dimensional black hole or black ring. Generic supersymmetric solutions in six-dimensions are expected to have a much richer structure than their five-dimensional counterparts, not only because such solutions can depend upon the compactification directions but also because we expect to find genuinely new, smooth supersymmetric solitons, like the superstratum [29].

We have shown that the system of equations underlying all supersymmetric solutions in this six-dimensional theory reduces to a linear system of equations after one has made the first step of laying down some of the geometric elements of the metric. Specifically, the six-dimensional metric must have the form (2.1) and the four-dimensional base is required to be almost hyper-Kähler with 2-forms satisfying (2.4). On this base manifold one must choose a vector field, β, whose field strength is self-dual, (2.14), and that satisfies the differential constraint (2.5). This vector field is also part of the five-dimensional spatial geometry because it determines how the S^1 is fibered over the spatial base. The choices of the base metric and the vector field, β, represent the only non-linearities in the system of equations. After this, all of the electric potentials, warp factors, magnetic fluxes and the angular momentum vector are determined by linear systems of equations. Moreover, if β is taken to be independent of the fiber coordinate, v, then it is simply a self-dual $U(1)$ gauge field and its equation of motion is also linear.

To summarize the process in more detail, once one has chosen an almost hyper-Kähler base metric on the four manifold and has fixed β, one uses the background to construct $\hat{\psi}$ as defined in (2.8). One then solves the coupled linear systems (2.28), (2.29) and (2.30) for Z_1 and Θ_2 and (2.27), (2.31) and (2.32) for Z_2 and Θ_1. The functions H and ϕ are then obtained from (2.26) and the forms K and $\mathcal{G}$ are obtained from (2.25). With these known sources, β and the base metric, equations (2.39) and (2.40) are *linear* in $\mathcal{F}$ and ω. As with all linear systems there are choices of homogeneous solutions and these correspond to sources for the fluxes or must be chosen to make the metric suitably regular and remove closed time-like curves.

It would, of course, be very nice to have some greater control over the general class of solutions, and have a more systematic understanding of the starting point of this linear system: the almost hyper-Kähler metrics with self-dual Maxwell fields defined by a vector field, β, satisfying (2.5). The self-duality equation for $D\beta$ is non-linear in β because of the definition (2.13), however if one expands in Fourier modes one can view this as Euclidean self-dual Yang-Mills where the structure constants are those of the (classical) Virasoro algebra. It may therefore be possible to solve this in a straightforward manner using some generalization of the ADHM construction. Alternatively, there might be an interesting classification of almost hyper-Kähler metrics and vector fields β if one assumes a $U(1)$ isometry on the base and that the almost hyper-Kähler forms are invariant under this action.

From past experience we expect our key observation of linearity to make the future construction of six-dimensional BPS solutions far simpler and to enable the construction of whole new classes of solutions through the superposition of others. In addition, the linear structure will also make the analysis of the moduli space of solutions a much more straightforward process and enable a much simpler analysis of BPS fluctuations around a given background. While the analysis of the generic almost hyper-Kähler base and the non-linear equation for β is, as yet, unsolved, it is worth remembering the observation made in Section 5: There are already going to be large new classes solutions that start from a general hyper-Kähler base and that are completely determined by a linear system on that base.

Apart from the fact that the classification of six-dimensional BPS solutions is interesting, our

primary motivation for dissecting the system of BPS equations is to look for new, smooth BPS solutions and particularly for the superstratum [29]. Here we have made some important steps in this direction. We have reduced the system of BPS equations to a far more manageable form and we have constructed a new class of three-charge solutions that have undergone the first of the two supertube transitions that will lead to the superstratum. We leave this the completion of this process for future work.

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